

# Gram matrix of a Laguerre model: application to model reduction of irrational transfer function

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An efficient method for evaluating inner products of repeated integrals and derivatives of a function described by a Laguerre model is presented. An immediate and promising application is model reduction of infinite dimensional transfer functions.

## 1. Introduction

The Gram matrix  $\Psi$  of a system is very useful in system identification, power density spectrum modeling and model reduction. The Gram matrix contains elements that are inner products of repeated integrals and/or derivatives of the impulse response of the system. In model reduction applications, the elements of this matrix  $\Psi$  have to be computed from the mathematical model of the original system. Note that only stable systems will be considered.

Let  $\Omega = \{f_1, f_2, \dots, f_q, \dots, f_{r+1}\}$  denote a set of contiguous real functions where

$$\begin{aligned} f_q(t) &\triangleq f(t), \\ f_{i+1}(t) &\triangleq \frac{df_i(t)}{dt} \quad (i = q, q+1, \dots, r), \\ f_{i-1}(t) &\triangleq \int_{-\infty}^t f_i(\tau) d\tau \quad (i = q, q-1, \dots, 2), \end{aligned} \quad (1)$$

$f(t)$  being the impulse response of the system with transfer function  $F(s)$ . Clearly, the computation of the Gram matrix involves evaluation of inner products of real functions

$$\psi_{i,j} \triangleq \langle f_i, f_j \rangle = \int_0^{\infty} f_i(t) f_j(t) dt. \quad (2)$$

Derivative and integral operators defined in (1) preserve natural frequencies or poles in the

Laplace domain, therefore  $\Omega$  constitutes an efficient set of approximation functions for determining a  $r$ -order reduced denominator. Moreover, during model-reduction procedure, integral operator will slightly emphasize the lower frequencies, when derivative operator will slightly emphasize the higher frequencies.

Usually  $f(t)$  will be the impulse response of the system, but assuming that  $f(t)$  is a well-behaved square-integrable function with Laplace transform  $F(s)$ , other responses of the system could be considered (for example the step response shifted of its final value).

When systems are described by rational transfer functions, integrals defined by (2) can be efficiently computed in the frequency domain [1–3]. Here we consider systems described by Laguerre models. Indeed the Laguerre functions have shown their large potential in numerous applications as in signal analysis and parameter identification [4], system identification [5,6], approximation of finite or infinite-dimensional system [7,8], numerical inversion of the Laplace transform [9–12], industrial control [13]... This is more particularly in the context of the numerical inversion of infinite-dimensional Laplace transfer functions that we have based our works. That restrictive choice does not withdraw anything from the generality of the presented results.

Let  $F(s)$  denote the transfer function of the system described by a Laguerre series

$$F(s) = \sum_{n=0}^{\infty} c_n \Phi_n(s) \quad (3)$$

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where  $\{c_n\}_{n \geq 0}$  is the Laguerre spectrum of the system,  $\Phi_n(s)$  are the Laplace transforms of the Laguerre functions defined by

$$\Phi_n(s) \triangleq \frac{\sqrt{2\alpha}}{s + \alpha} \left( \frac{s - \alpha}{s + \alpha} \right)^n \quad (4)$$

for  $n = 0, 1, 2, \dots$  and  $(-\alpha)$  is a real pole satisfying  $\alpha > 0$ . In practice the Laguerre series (3) will be truncated and the Laguerre coefficients  $c_n$  will be evaluated by a well-known technique based on discrete Fourier transform computation [9–12].

Now, let  $F_i(s)$  denote the Laplace transform of the function  $f_i(t)$  and  $\{c_{i,n}\}_{n \geq 0}$  its Laguerre spectrum. As the Laguerre functions form an orthonormal basis, the inner products defined by (2) can be written in Laguerre spectral domain as

$$\psi_{i,j} = \sum_{n=0}^{\infty} c_{i,n} c_{j,n}. \quad (5)$$

Integrals defined by (2) are thus replaced by sums, in practice finite, that are simpler to calculate and numerically more robust. Before evaluating these sums, Laguerre spectra of the repeated derivatives and integrals of  $f(t)$  must be computed.

## 2. Derivatives and integrals computation

A well-known relation binding the Laplace transform and the z-transform of the Laguerre spectrum  $\{c_n\}_{n \geq 0}$  is deduced by substituting (4) in (3), then carrying out the transformation  $s \rightarrow \alpha \frac{z+1}{z-1}$ , one obtains

$$\begin{aligned} Z\{c_n\}_{n \geq 0} &\triangleq \sum_{n=0}^{\infty} c_n z^{-n} \\ &= \sqrt{2\alpha} \frac{z}{z-1} F\left(\alpha \frac{z+1}{z-1}\right) \end{aligned} \quad (6)$$

where  $Z$  denotes the z-transform operator. This useful relation will enable us to determine the Laguerre spectra of repeated integrals and derivatives of the function  $f(t)$ .

Let us first consider the derivative operator defined by (1), in Laplace domain it yields

$$F_{i+1}(s) = sF_i(s) - f_i(+0). \quad (7)$$

Substitution of (7) in (6) leads to

$$\begin{aligned} Z\{c_{i+1,n}\}_{n \geq 0} &= \\ \alpha \frac{z+1}{z-1} Z\{c_{i,n}\}_{n \geq 0} - \sqrt{2\alpha} \frac{z}{z-1} f_i(+0) \end{aligned}$$

which yields the recurrence relation

$$\begin{cases} c_{i+1,0} = \alpha c_{i,0} - \sqrt{2\alpha} f_i(+0) \\ c_{i+1,n} = \alpha(c_{i,n} + c_{i,n-1}) + c_{i+1,n-1} \end{cases} \quad (8)$$

for  $n = 1, 2, 3, \dots$  and  $i = q, q+1, \dots, r$ . When the initial value  $f_i(+0)$  is not known it can be computed as follows

$$f_i(+0) = \lim_{s \rightarrow \infty} sF_i(s) = \sqrt{2\alpha} \sum_{n=0}^{\infty} c_{i,n}. \quad (9)$$

In practice, the Laguerre series is usually truncated at order  $N$ . Therefore, and more particularly when the initial value  $f_i(+0)$  is not readily available, an approximation of the coefficients of the various Laguerre spectra can be computed as follows

$$\begin{cases} \tilde{c}_{i+1,N-1} = -\alpha \tilde{c}_{i,N-1} \\ \tilde{c}_{i+1,n-1} = \tilde{c}_{i+1,n} - \alpha(\tilde{c}_{i,n} + \tilde{c}_{i,n-1}) \end{cases} \quad (10)$$

for  $n = N-1, N-2, \dots, 1$ ,  $i = q, q+1, \dots, r$  and where the starting point of the computation is  $\tilde{c}_{q,n} = c_{q,n}$ . The approximated Laguerre series of  $F_i(s)$  is then given by

$$\tilde{F}_i(s) = \sum_{n=0}^{N-1} \tilde{c}_{i,n} \Phi_n(s).$$

An approximated value of  $f_i(+0)$  can then be evaluated using (8) or (9) i.e.

$$\tilde{f}_i(+0) = \sqrt{2\alpha} \sum_{n=0}^{N-1} \tilde{c}_{i,n} = \frac{1}{\sqrt{2\alpha}} (\alpha \tilde{c}_{i,0} - \tilde{c}_{i+1,0}). \quad (11)$$

Note that the procedure (10) gives a set of functions verifying the strict equality

$$\tilde{f}_{i+1}(t) = \frac{d\tilde{f}_i(t)}{dt}$$

Integral and derivative operators defined in (1) are reciprocal, therefore the recurrence relations binding the Laguerre spectra of the repeated integrals of the function  $f(t)$  are readily obtained from (8)

$$\begin{cases} c_{i-1,0} = \frac{1}{\alpha} (c_{i,0} + \sqrt{2\alpha} f_{i-1}(+0)) \\ c_{i-1,n} = \frac{1}{\alpha} (c_{i,n} - c_{i,n-1}) - c_{i-1,n-1} \end{cases} \quad (12)$$

for  $n = 1, 2, 3, \dots$  and  $i = q, q-1, \dots, 2$ . Dealing with truncated Laguerre models, and more particularly when the initial value  $f_{i-1}(+0)$  is not readily available, an approximation of the Laguerre coefficients can be recursively computed in the following way

$$\begin{cases} \tilde{c}_{i-1,N-1} = -\frac{1}{\alpha} \tilde{c}_{i,N-1} \\ \tilde{c}_{i-1,n-1} = \frac{1}{\alpha} (\tilde{c}_{i,n} - \tilde{c}_{i,n-1}) - \tilde{c}_{i-1,n} \end{cases} \quad (13)$$

for  $n = N-1, N-2, \dots, 1$ ,  $i = q, q-1, \dots, 2$  and where the starting point of the computation is  $\tilde{c}_{q,n} = c_{q,n}$ . Note that the procedure (13) gives a set of functions verifying the strict equality

$$\tilde{f}_{i-1}(t) = \int_{-\infty}^t \tilde{f}_i(\tau) d\tau.$$

### 3. Gram matrix computation

The Gram matrix of repeated integral and derivative functions possesses interesting properties [14,15], that allow reduction of the computing cost of the inner products (5). Indeed integration by parts of (2) yields

$$\psi_{i,j} = [f_{i-1}(t) f_j(t)]_0^\infty - \int_0^\infty f_{i-1}(t) f_{j+1}(t) dt.$$

Because  $f(t)$  is square integrable by assumption, this relation simplifies to

$$\psi_{i,j} = -f_{i-1}(+0) f_j(+0) - \psi_{i-1,j+1} \quad (14)$$

and, in the particular case  $j = i-1$  one obtains

$$\psi_{i,i-1} = -\frac{1}{2} f_{i-1}^2(+0). \quad (15)$$

The calculation algorithm of the Gram matrix is then the following:

1. Compute the Laguerre spectra of successive integrals and derivatives or their approximations using (12) or (13) and (8) or (10).
2. Compute the elements of the main diagonal of the Gram matrix using (5) or in practice the following truncated sums  $\psi_{i,i} \approx \sum_{n=0}^{N-1} c_{i,n}^2$  or  $\psi_{i,i} \approx \sum_{n=0}^{N-1} \tilde{c}_{i,n}^2$ .
3. Compute the elements of the first secondary diagonal using (15).
4. Recursively compute the other elements of the lower triangular part of the Gram matrix using (14).
5. The upper triangular part of the matrix is readily obtained using the symmetry property  $\psi_{i,j} = \psi_{j,i}$ .

Note that if initial values are not readily available, it is then possible to evaluate them using (11).

**Remark:** It will be noted that the described procedures (8) and (12) or (10) and (13) give the Laguerre spectra of repeated strict derivatives and integrals. Therefore, in model reduction application, the reduced model of  $F(s)$  is provably stable by a Lyapunov method if the Gram matrix is computed on  $\Omega$  or on  $\tilde{\Omega} = \{\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_q, \dots, \tilde{f}_{r+1}\}$  and if the model reduction procedure is based on the approximation of either first or last function of the set by the others [16].

### 4. Example

To illustrate the utility of the method we present a typical application of model reduction of an infinite-dimensional system. The system considered is the celebrated underwater cable whose irrational transfer function is [17,18]

$$G(s) = e^{-\sqrt{K}s} \quad (\text{with } K = 1).$$

From  $G(s)$ , the first  $N = 100$  coefficients of a Laguerre model with  $\alpha = 2.42$  is computed using a well-known technique [9–12] based on discrete Fourier transform computation. For ease

$$\tilde{G}(s) = \frac{-0.0261s^5 + 7.41s^4 - 1048s^3 + 63397s^2 + 92844s + 6326}{s^6 + 98.3s^5 + 4568s^4 + 67984s^3 + 227350s^2 + 132925s + 6686}$$

of experimentation, however, we have coded our own version of the algorithm using a discrete cosine transform. The (not crucial) choice  $\alpha = 2.42$  for the computation of the Laguerre spectrum of  $G(s)$  is obtained using the optimization method described in [19]. In order to derive reduced-order models taking the form of rational functions of order  $r = 6$ , Gram matrices of size  $(r + 1) \times (r + 1)$  are calculated. We used the method described in [14,15] to derive some reduced models corresponding to the values  $q = 1, 2, \dots, r + 1$ . We retained the reduced model that minimizes the relative quadratic error defined by

$$\varepsilon \triangleq \frac{\int_0^\infty [g(t) - \tilde{g}(t)]^2 dt}{\int_0^\infty [g(t)]^2 dt}$$

where  $\tilde{g}(t)$  is the impulse response of the 6th-order model and  $g(t)$  is the exact impulse response of  $G(s)$  given by

$$g(t) = \frac{1}{2K\sqrt{\pi}} \left( \frac{K}{t} \right)^{\frac{3}{2}} e^{-\frac{K}{4t}}.$$

The selected reduced model  $\tilde{G}(s)$ , given at the top of the page, is based on the Gram matrix constructed with the scalar products of the Laguerre models of  $G(s)$ , its 4 first derivatives and its 2 first integrals (i.e.  $q = 3$ ). The corresponding quadratic error  $\varepsilon = 5.24 \times 10^{-4}$  confirms the great accuracy of the reduced model.

## 5. Conclusion

An efficient procedure for computing the Gram matrix of repeated derivatives and integrals of a function described by a Laguerre model has been presented. Integrals defining the required inner products are replaced by sums (finite in practice) allowing a calculation numerically more convenient. The deduced recurrence relations yield

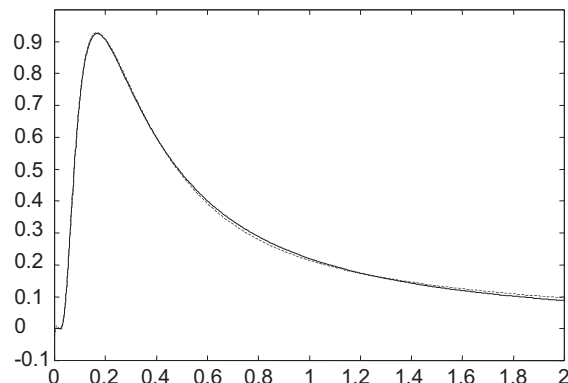


Figure 1. Impulse responses of the original system and of its 6th-order model (dashed line)

a drastic reduction of the computation cost of the Gram matrix. Presented work takes part of model reduction of infinite dimensional systems. An illustrative example has shown the efficiency of the procedure.

## REFERENCES

1. L. C. Calvez, P. Vilbé, P. Bréhonnet, Evaluation of scalar products of repeated integrals of a function with rational Laplace transform, *Electronics Letters* 24 (11) (1988) 658–659.
2. L. C. Calvez, P. Vilbé, M. Sévellec, Efficient evaluation of model-reduction related integrals via polynomial arithmetic, *Electronics Letters* 28 (7) (1992) 659–660.
3. T. N. Lucas, Evaluation of scalar products of repeated integrals by Routh algorithm, *Electronics Letters* 24 (20) (1988) 1290–1291.
4. P. R. Clement, Laguerre functions in signal analysis and parameter identification, *Journal of the Franklin Institute* 313 (2) (1982) 85–95.
5. C. T. Chou, M. Verhaegen, R. Johansson,

- Continuous-time identification of SISO systems using Laguerre functions, *IEEE Trans. Signal Processing* 47 (2) (1999) 349–362.
6. B. Wahlberg, System identification using Laguerre models, *IEEE Trans. Automatic Control* 36 (5) (1991) 551–562.
  7. P. M. Mäkilä, Approximation of stable systems by Laguerre filters, *Automatica* 26 (2) (1990) 333–345.
  8. P. M. Mäkilä, Laguerre series approximation of infinite dimensional systems, *Automatica* 26 (9) (1990) 985–995.
  9. J. Abate, G. L. Choudhury, W. Whitt, On the Laguerre method for numerically inverting Laplace transforms, *INFORMS Journal on Computing* 8 (4) (1996) 413–427.
  10. F. Tricomi, Trasformazione di Laplace e polinomi di Laguerre, *R. C. Accad. Nat. dei Lincei* 21 (1935) 232–239.
  11. E. E. Ward, The calculation of transients in dynamical systems, *Proc. Camb. Phil. Soc.* 50 (1954) 49–59.
  12. W. T. Weeks, Numerical inversion of Laplace transforms using Laguerre functions, *J. ACM* 13 (1966) 419–426.
  13. G. A. Dumont, C. C. Zervos, G. Pageau, Laguerre-based adaptive control of pH in an industrial bleach plant extraction stage, *Automatica* 26 (4) (1990) 781–787.
  14. M. Sévellec, Réduction de l'ordre des systèmes à fonctions de transfert rationnel, Thèse de Doctorat, Brest Univ., Brest, France, 1993.
  15. P. Vilbé, L. C. Calvez, M. Sévellec, Suboptimal model reduction via least-square approximation of time-response by its derivatives and integrals, *Electronics Letters* 28 (2) (1992) 174–175.
  16. H. Nagaoka, Mullis-Roberts-type approximation for continuous-time linear systems, *Electronics and Communications in Japan* 70 (10) (1987) part 1.
  17. C. Hsu, D. Hou, Linear approximation of fractional transfer functions of distributed parameter systems, *Electronics Letters* 26 (15) (1990) 1211–1213.
  18. R. Morvan, Modélisation de circuits et systèmes de dimension infinie, Thèse de Doctorat, Brest Univ., Brest, France, 2000.
  19. N. Tanguy, P. Vilbé, L. C. Calvez, Optimum choice of free parameter in orthonormal approximations, *IEEE Trans. Automatic Control* 40 (10) (1995) 1811–1813.